Department of Physics
Kyoto University

KUNS 1500

Naked singularity formation in the collapse of a spherical cloud of counterrotating particles

Tomohiro Harada *, Hideo Iguchi †, and Ken-ichi Nakao ‡
Department of Physics, Kyoto University, Kyoto 606-8502, Japan

We investigate the collapse of a spherical cloud of counter-rotating particles. An explicit solution for metric functions is given using an elliptic integral. If the specific angular momentum $L(r)=O(r^2)$ at $r\to 0$, no central singularity occurs. With $L(r)$ like that, there is a finite region around the center that bounces. On the other hand, if the order of $L(r)$ is higher than that, a central singularity occurs. In marginally bound collapse with $L(r)=4F(r)$, a naked singularity occurs, where $F(r)$ is the Misner-Sharp mass. The solution for this case is expressed by elementary functions. For $4<L/F<\infty$ at $r\to 0$, there is a finite region around the center that bounces and a naked singularity occurs. For $0\le L/F<4$ at $r\to 0$, there is no such region. The results suggest that rotation may play a crucial role on the final fate of collapse.

PACS numbers: 04.20.Dw, 04.20.Jb, 04.40.-b

The final fate of gravitational collapse is an important problem of relativistic astrophysics and gravitational physics. A black hole is usually considered as the final state of generic gravitational collapse, and its properties have been investigated [1]. In particular, a cosmic censorship hypothesis [2] is a critical assumption of theorems on black holes. For example, singularity theorems [1] with this hypothesis predict the existence of black holes. Therefore, it is very important to know the behavior of collapse of various kinds of matter in order to obtain the overall picture about the general nature of gravitational collapse. For example, it is well known that spherically symmetric dust collapse from generic initial data results in a naked singularity [3–6]. This system is rather tractable because of the existence of an explicit expression, i.e., the Lemaître-Tolman-Bondi (LTB) solution. The assumption of dust matter will be considered to be rather unrealistic. In fact, it is obvious that the effect of pressure should be taken into account because the formation of a naked singularity in the LTB solution results in blow up of the energy density. Introducing tangential pressure alone does not lose the merit because this system is also given by an explicit integral [7]. Singh and Witten [8] considered tangential pressure proportional to the energy density and found that in collapse from rest there is a finite region near the center which expands outwards if the tangential pressure is positive. A spherical cloud of counter-rotating particles is an example with tangential pressure but no radial pressure. This system was considered by Datta [9], Bondi [10] and Evans [11]. However, the causal structure of the space-time was not investigated and this is one of our interests.

Using comoving coordinates, a spherically symmetric space-time describing a matter with no radial pressure is given by

$$ds^2=-e^{2\nu}dt^2+\frac{(R')^2h^2}{1+f}dr^2+R^2(d\theta^2+\sin^2\theta d\phi^2), \tag{1}$$

where $h=h(r,R)$ gives a relation between the energy density $\epsilon(r,t)\equiv -T_t^t$ and the tangential pressure $\Pi(r,t)\equiv T_\theta^\theta=T_\phi^\phi$ as

$$\Pi=-\frac{R}{2h}\frac{\partial h}{\partial R}\epsilon, \tag{2}$$

* Electronic address: harada@tap.scphys.kyoto-u.ac.jp
† Electronic address: iguchi@tap.scphys.kyoto-u.ac.jp
‡ Electronic address: nakao@tap.scphys.kyoto-u.ac.jp

and $\nu = \nu(r,t)$ and $R = R(r,t)$ satisfy the following set of coupled partially derivative equations:

$$\nu' = -\frac{1}{h}\frac{\partial h}{\partial R}R'. \tag{3}$$

$$\dot{R}^2 e^{-2\nu} = -1 + \frac{2F}{R} + \frac{1+f}{h^2}. \tag{4}$$

The prime and overdot denote the derivative with respect ot r and t, respectively. The energy density is given by

$$\epsilon = \frac{F'}{4\pi R^2 R'} \tag{5}$$

The arbitrary functions $F(r)$ and $f(r) > -1$ are the conserved Misner-Sharp mass and the specific energy, respectively. The function $h(r,R) > 0$ has a meaning of the internal elastic energy per volume and the dust limit is given by $h = 1$ [12]. From regularity at the center, $f(0) = h^2(0,0) - 1$, $R(0,t) = 0$, $|\nu(0,t)| < \infty$ and $F(r)/r^3 < \infty$ at $r \to 0$. Note that we can set $f(0) = h^2(0,0) - 1 = 0$ because only the ratio of h^2 to $1+f$ is meaningful. The solution can be matched with the Schwarzschild space-time at an arbitrary radius $r = r_b$ if we identify the Schwarzschild mass parameter M with $F(r_b)$. Assuming that R is initially a monotonically increasing function of r and rescaling the radial coordinate r, we identify r with the circumferential radius R on the initial space-like hypersurface $t = 0$. Here we search the location of an apparent horizon in a collapsing phase. Along a future-directed outgoing null geodesic,

$$\frac{dR}{dt} = \dot{R} + R'\frac{dr}{dt} = e^{\nu}\left[-\sqrt{-1 + \frac{2F}{R} + \frac{1+f}{h^2}} + \sqrt{\frac{1+f}{h^2}}\right]. \tag{6}$$

Therefore $R = 2F$ is an apparent horizon, $0 \le R < 2F$ is a trapped region, and $2F < R$ is an untrapped region.

Tangential pressure for a spherical cloud of counter-rotating particles is given by [9–11]

$$\Pi = \frac{1}{2}\frac{L^2(r)}{R^2 + L^2(r)}\epsilon, \tag{7}$$

where $L(r)$ is the specific angular momentum. Then, the function h is given by

$$h^2 = \frac{R^2 + L^2}{R^2}. \tag{8}$$

Because of the coupling of Eqs. (3) and (4), the solution given above is not explicit. Here, according to the procedure of Magli [7], we introduce the mass-area coordinates and give an explicit form for the general solution. The space-time is written in these coordinates as

$$ds^2 = -Adm^2 - 2BdRdm - CdR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{9}$$

where m agrees with the Misner-Sharp mass $F(r)$. Solving the Einstein equations, we obtain the metric functions $A(m,R)$, $B(m,R)$ and $C(m,R)$ as follows:

$$A = H(1 - \frac{2m}{R}), \tag{10}$$

$$B = -\frac{ER\sqrt{H}}{|u|\sqrt{R^2 + L^2}}, \tag{11}$$

$$C = \frac{1}{u^2}, \tag{12}$$

$$u \equiv \frac{dR}{d\tau} = \pm\sqrt{-1 + \frac{2m}{R} + \frac{E^2R^2}{R^2 + L^2}}, \tag{13}$$

$$\begin{aligned}\sqrt{H(m,R)} = {} & \frac{(F^{-1})_{,m}\sqrt{(F^{-1})^2 + L^2}}{EF^{-1}}\left(\frac{E^2(F^{-1})^2}{(F^{-1})^2 + L^2} - 1 + \frac{2m}{F^{-1}}\right)^{-1/2} \\ & + \int_{F^{-1}}^{R}\frac{\sqrt{x^2 + L^2}}{x^2 E}\left[1 + \frac{x^3E^2}{2(x^2 + L^2)}\left(\frac{(E^2)_{,m}}{E^2} - \frac{(L^2)_{,m}}{x^2 + L^2}\right)\right] \\ & \cdot\left(-1 + \frac{2m}{x} + \frac{E^2x^2}{x^2 + L^2}\right)^{-3/2} dx,\end{aligned} \tag{14}$$

where τ is a proper time of a comoving observer at a mass shell labeled by r. The upper and lower signs correspond to expanding and collapsing phases, respectively. F^{-1} is the inverse function of F and the existence of it is guaranteed by the positivity of the energy density, which hereafter we assume. We have defined $E^2(m) \equiv 1 + f(F^{-1}(m))$. In order to determine an arbitrary function which appears in the expression of $\sqrt{H}$, we have set $R = F^{-1}(m)$ on the initial space-like hypersurface, which corresponds to $R(r, 0) = r$. We have rewritten $L(F^{-1}(m))$ as $L(m)$. The energy density is given by

$$\epsilon = \frac{\sqrt{R^2 + L^2}}{4\pi |u| R^3 E \sqrt{H}}. \tag{15}$$

Then, a shell-crossing singularity occurs when $u\sqrt{H} = 0$. If a shell-crossing occurs, the coordinate system breaks down after that. The integral in the expression of $\sqrt{H}$ is reduced to

$$\int_{F^{-1}}^{R} \frac{2(x^2+L^2)^2 + x^3[(E^2)_{,m}(x^2+L^2) - (L^2)_{,m}E^2]}{2E[(E^2-1)x^3 + 2mx^2 - L^2x + 2mL^2]\sqrt{(E^2-1)x^4 + 2mx^3 - L^2x^2 + 2mL^2x}} dx. \tag{16}$$

Therefore we find that the general solution is expressed by an elliptic integral.

If $L(m) = 0$, the solution is reduced to the LTB solution in the mass-area coordinates. If $E(m) = 1$ (which is called marginally bound collapse) and $L(m) = 4m$, the integral is expressed by elementary functions as follows:

$$A = H(1 - \frac{2m}{R}), \tag{17}$$

$$B = -\frac{R}{|R-4m|}\sqrt{\frac{RH}{2m}}, \tag{18}$$

$$C = \frac{R(R^2+16m^2)}{2m(R-4m)^2}, \tag{19}$$

$$u = \pm|R-4m|\sqrt{\frac{2m}{R(R^2+16m^2)}}, \tag{20}$$

$$\begin{aligned}\sqrt{H} = &\frac{(F^{-1})_{,m}((F^{-1})^2 + 16m^2)}{|F^{-1} - 4m|\sqrt{2mF^{-1}}} \\ &+ \mathrm{sign}(F^{-1} - 4m)\left[\left(\frac{R^2 - 16mR + 144m^2}{3\sqrt{2}m(R-4m)}\sqrt{\frac{R}{m}} + 4\sqrt{2}\ln\frac{\sqrt{R} + 2\sqrt{m}}{|\sqrt{R} - 2\sqrt{m}|}\right)\right. \\ &\left. - \left(\frac{(F^{-1})^2 - 16mF^{-1} + 144m^2}{3\sqrt{2}m(F^{-1}-4m)}\sqrt{\frac{F^{-1}}{m}} + 4\sqrt{2}\ln\frac{\sqrt{F^{-1}} + 2\sqrt{m}}{|\sqrt{F^{-1}} - 2\sqrt{m}|}\right)\right].\end{aligned} \tag{21}$$

Now that we have obtained the explicit expression for the metric functions, we investigate the singularity that may occur in the collapse of a spherical cloud of counter-rotating particles. Since shell-crossing singularities are considered to be gravitationally weak, we concentrate on shell-focusing singularities which is defined by $R = 0$. Furthermore, no light ray can emmanate from non-central ($r > 0$) shell-focusing singularities because $R = 0 < 2F$ for $r > 0$ and hence it is covered [4]. From the above discussion, it turns out to be sufficient to consider a central ($r = 0$) shell-focusing singularity in order to discuss whether or not a naked singularity exists. The motion of each shell of $r > 0$ is described by Eq. (4), i.e.,

$$\left(\frac{dR}{d\tau}\right)^2 = -1 + \frac{2F}{R} + \frac{(1+f)R^2}{R^2 + L^2}. \tag{22}$$

By investigating the shape of the effective potential

$$V(R; r) \equiv -\frac{2F}{R} + 1 - \frac{(1+f)R^2}{R^2 + L^2}, \tag{23}$$

we can get qualitative understanding about the motion of each shell of $r > 0$.

Proceeding to deal with the symmetric center $r = 0$, we take the analyticity there into consideration. We assume that the metric functions and the function $h^2(r, R)$ are C^∞ class at least in the neighborhood of the center $r = 0$

before encountering a central singularity. From this assumption, the metric variables in the comoving coordinates are expanded as

$$\nu(r,t) = \nu_0(t) + \nu_2(t)r^2 + \nu_4(t)r^4 + \cdots, \tag{24}$$

$$R(r,t) = R_1(t)r + R_3(t)r^3 + R_5(t)r^5 + \cdots, \tag{25}$$

and we can set $\nu_0(t) = 0$ by using the rescaling freedom of the time coordinate. Then, from Eqs. (3) and (4), the arbitrary functions $F(r)$, $f(r)$ and $L^2(r)$ should be expanded as

$$F(r) = F_3 r^3 + F_5 r^5 + \cdots, \tag{26}$$

$$f(r) = f_2 r^2 + f_4 r^4 + \cdots, \tag{27}$$

$$L^2(r) = L_4 r^4 + L_6 r^6 + \cdots, \tag{28}$$

and, from Eq. (5), the energy density should be expanded as

$$\epsilon(r,t) = \epsilon_0(t) + \epsilon_2(t)r^2 + \epsilon_4(t)r^4 + \cdots, \tag{29}$$

and then

$$\epsilon(0,t) = \epsilon_0(t) = \frac{3F_3}{4\pi R_1(t)^3}. \tag{30}$$

Observing the lowest order of Eq. (4), the time development of $R_1(t)$ is given by

$$\left(\frac{dR_1}{dt}\right)^2 = \frac{2F_3}{R_1} + f_2 - \frac{L_4}{R_1^2}. \tag{31}$$

Since $L^2(r) \geq 0$, $L_4 \geq 0$.

First we consider the case of $L_4 > 0$. From Eq. (31) we find that $R_1(t)$ cannot vanish. This result was firstly shown by Evans [11]. Therefore no central singularity occurs for $L_4 > 0$. Next we examine the motion of the shell $r > 0$. From Eq. (23), allowed regions for a given r are obtained by

$$g(R;r) \equiv -fR^3 - 2FR^2 + L^2R - 2FL^2 \leq 0. \tag{32}$$

For sufficiently small $r > 0$,

$$g(R = 0; r) \approx -2FL^2 < 0, \tag{33}$$

$$g(R = 4F; r) \approx 2FL^2 > 0, \tag{34}$$

where "$\approx$" means the equality up to the lowest order. Considering that $g(R;r)$ is a cubic function of R, the allowed regions are given by

$$0 \leq R \leq R_1, \quad R_2 \leq R, \quad (\text{for} \quad f \geq 0), \tag{35}$$

$$0 \leq R \leq R_1, \quad R_2 \leq R \leq R_3, \quad (\text{for} \quad f < 0), \tag{36}$$

where the following inequality is satisfied

$$0 < R_1 < 4F < R_2. \tag{37}$$

Since $F = O(r^3)$, $R(r, t = 0) = r$ cannot be in the inner allowed region $0 \leq R \leq R_1$. This means that $R(r,t)$ must be in the outer allowed region at $t = 0$. Therefore we conclude that the region around $r = 0$, which was initially in a collapsing phase, necessarily experiences a bounce and begins to expand. The motion after that is an eternal expansion for $f \geq 0$ or oscillations for $f < 0$. This implies that, since $R > R_2 > 4F > 2F$, the region around $r = 0$ is untrapped. We summarize this case by no central singularity, no apparent horizon and bounce of the region around the center.

We proceed to the case of $L_4 = 0$. From Eq. (31), we find that the initially collapsing cloud inevitably form a central shell-focusing singularity after a finite proper time. In order to see whether this central singularity is naked or covered, we examine the motion of the region around the center. Here we define

$$D \equiv \lim_{r\to 0} \frac{L}{F} = L_6^{1/2}/F_3. \tag{38}$$

For $D > 4$, for sufficiently small $r > 0$,

$$g(R = 0; r) \approx -2FL^2 < 0, \tag{39}$$

$$g(R = \frac{D^2}{4}F; r) \approx \frac{D^2(D^2 - 16)}{8}F^3 > 0. \tag{40}$$

Then, the allowed regions are

$$0 \le R \le R_1, \quad R_2 \le R \quad (\text{for} \quad f \ge 0), \tag{41}$$

$$0 \le R \le R_1, \quad R_2 \le R \le R_3 \quad (\text{for} \quad f < 0), \tag{42}$$

where the following inequality is satisfied:

$$0 < R_1 < \frac{D^2}{4}F < R_2. \tag{43}$$

In the same way as for the case of $L_4 > 0$, we conclude that the region around $r = 0$, which was initially in a collapsing phase, necessarily experiences a bounce and begins to expand. The motion after that is an eternal expansion for $f \ge 0$ or oscillations for $f < 0$. This implies that, since $R > R_2 > 4F > 2F$, the region around $r = 0$ is untrapped. Therefore the central singularity is naked. Since the space-time can be matched to the Schwarzschild space-time at an arbitrary radius $r = r_b$, we can construct the space-time with a globally naked singularity. The results given above does not depend on details of initial density distribution.

For $D = 4$, the behavior depends on the higher order terms. Here we present the critical and most interesting case in which $f(r) = 0$ and $L(r) = 4F(r)$. For this case, the metric functions are exactly solved and expressed by elementary functions as seen in Eqs. (17)- (21). In this case the effective potential is given by

$$V = -\frac{2F(R - 4F)^2}{R(16F^2 + R^2)}. \tag{44}$$

On the initial space-like hypersurface, regularity requires $R(r, t = 0) = r > 4F = O(r^3)$ in a sufficiently small but finite region around $r = 0$. Then each initially collapsing shell of $r > 0$ approaches $R = 4F$. From Eq. (22), the behavior of this approach is as

$$R - 4F \propto \exp(-\frac{\tau}{8F}). \tag{45}$$

This behavior means that R approaches $4F$ asymptotically. This behavior is also the case in an expanding phase for $R < 4F$, and therefore the sign of $R - 4F$ does not change. That is why we have chosen the sign as seen in Eqs. (17)-(21). Then, in the region around the center $r = 0$, the collapse suffers the continuing angular momentum braking. Since $R > 4F > 2F$, the region around the center is untrapped eternally. Therefore the central singularity is naked and can be globally naked. Furthermore we prove that a shell crossing does not occur and the coordinate system is valid at least in the region around the center by showing that $\sqrt{H} > 0$. Each term in the first parenthesis in the bracket on the right hand side of Eq. (21) is positive, because $R > 4F$ in the region around the center. The contribution of the first term and the terms in the second parenthesis is expanded as

$$\frac{\sqrt{2}}{9}\frac{24F_3^2 - F_5}{F_3^{13/6}}m^{-1/3} + O(m^{1/3}), \tag{46}$$

where terms which is $O(m^{-1})$ cancel each other. Therefore, if $F_5 < 24F_3^2$, at least a sufficiently small region around the center is shell-crossing free, and we can construct a shell-crossing free solution by matching. This condition holds if $\epsilon(r, t = 0)$ is decreasing function of r. The nakedness of the central singularity is confirmed by examining the algebraic root equation given by Magli [7]. If there is a finite positive root x_0 for this equation for some $\alpha > 1/3$, the central singularity is naked. Using the exact solution (21), we obtain a finite positive root

$$x_0 = \left(\frac{24F_3^2 - F_5}{4\sqrt{2}F_3^{13/6}}\right)^{2/3}, \tag{47}$$

for $\alpha = 7/9$. Then, the future-directed outgoing null geodesic starting from the central singularity behaves as

$$R \approx 2x_0 m^{7/9} \approx 2x_0 F_3^{7/9} r^{7/3}. \tag{48}$$

For the case $0 \leq D < 4$, the collapse continues to a covered singularity for $r > 0$. In order to see whether the central singularity is naked or not, we have to examine the existence of positive and real roots of the root equation given in the mass-area coordinates by Magli [7] using the explicit form of the solution (10)-(14). Works for $0 \leq D < 4$ are now in progress.

In summary, we have investigated the final fate of the collapse of a spherical cloud of counter-rotating particles. We have presented an explicit expression for the metric functions and shown that this is given by an elliptic integral. For marginally bound collapse with angular momentum distribution $L(r) = 4F(r)$, we have succeeded to give an expression by elementary functions. The existence of the lowest order term in $L(r)$ that is allowed by regularity always guarantees regularity of the evolved center. On the other hand, we have shown that the absence of this term inevitably results in a central singularity in the collapse of the cloud. In the former case, the region around the center necessarily bounces to an expanding phase. In the latter case, if

$$\lim_{r\to 0} \frac{L}{F} > 4, \tag{49}$$

the region around the center $r = 0$ bounces to an expanding phase and the central singularity is naked and can be globally naked by matching to the Schwarzschild space-time. If $f(r) = 0$ and $L(r) = 4F(r)$, the region around the center slows down and approaches asymptotically to $R = 4F$ and the central singularity is also naked. If the limit in Eq. (49) is smaller than 4, the region around the center collapses to a space-like singularity. These results shows that *tangential pressure may undress the covered singularity*. In particular, rotation may induce the naked singularity formation. For example, a sufficiently small, marginally bound, and spherical ball of counter-rotating particles with $\epsilon(r, t = 0) =$ const and $L(r) = 4F(r)$ collapses to a naked singularity. Our results suggest that the effect of pressure is not negligible. Furthermore anisotropic velocity dispersion of gravitating particles and/or anisotropic pressure which may be realized by crystallization in a dense nuclear matter may play an important role in the final stage of collapse.

ACKNOWLEDGMENTS

We are grateful to T. Nakamura and D. Ida for helpful discussions. We are also grateful to H. Sato for his continuous encouragement. This work was partly supported by the Grant-in-Aid for Scientific Research Fellowship (No.9204) and for Creative Basic Research (No.09NP0801) from the Japanese Ministry of Education, Science, Sports and Culture.

---

[1] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, (Cambridge University Press, Cambridge, England) (1973).
[2] R. Penrose, "Singularities and Time Asymmetry", in *General Relativity, an Einstein Centenary Survey*, ed. S.W. Hawking and W. Israel, (Cambridge University Press, Cambridge, England), p581, (1979).
[3] D.M. Eardly and L. Smarr, Phys. Rev. D **19**, 2239, (1979).
[4] D. Christodoulou, Commun. Math. Phys. **93**, 171, (1984).
[5] P.S. Joshi and I.H. Dwivedi, Phys. Rev. D **47**, 5357, (1993).
[6] T.P. Singh and P.S. Joshi, Class. Quantum Grav. **13**, 559, (1996); S. Jhingan, P.S. Joshi and T.P. Singh, Class. Quantum Grav. **13**, 3057, (1996).
[7] G. Magli, gr-qc/9711082.
[8] T.P. Singh and L. Witten, Class. Quantum Grav. **14**, 3489, (1997).
[9] B.K. Datta, Gen. Relat. Grav. **1**, 19, (1970).
[10] H. Bondi, Gen. Relat. Grav. **2**, 321, (1971).
[11] A.B. Evans, Gen. Relat. Grav. **8**, 155, (1976).
[12] G. Magli, Class. Quantum Grav. **14**, 1937, (1997).